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Finite difference and Runge–Kutta methods for solving vibration problems

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Abstract. The vibration of a storey building can be modelled into a system of second order ordinary differential equations. If the number of floors of a building is large, then the result is a large scale system of second order ordinary differential equations. The large scale system is difficult to solve, and if it can be solved, the solution may not be accurate. Therefore, in this paper, we seek for accurate methods for solving vibration problems. We compare the performance of numerical finite difference and Runge–Kutta methods for solving large scale systems of second order ordinary differential equations. The finite difference methods include the forward and central differences. The Runge–Kutta methods include the Euler and Heun methods. Our research results show that the central finite difference and the Heun methods produce more accurate solutions than the forward finite difference and the Euler methods do.

1. Introduction

Vibration problems have been an interesting research area in physics, engineering, as well as applied mathematics. Modelling vibrations on the structural system, such as storey buildings, have the same principles as modelling vibrations on the spring-mass system (for more details see [1-6]). This is done by assuming that each floor of the building is a mass, and pillars of the building are considered as a spring which has a stiffness. Therefore, a vibration in storey buildings can be modelled into a system of second order ordinary differential equations.

Furthermore, if the floor of the building is large, then we can model it to a large scale system of second order ordinary differential equations [7-8]. This makes the system of ordinary differential equations difficult to solve [9]. In this case, we need some solving methods that produces accurate solutions.

Therefore, in this paper, we compare the performance of numerical finite difference and Runge–Kutta methods to solve large scale systems of second order ordinary differential equations. From our research results, we shall know which numerical method has a higher degree of accuracy, so it produces accurate solutions. We note that in engineering applications, vibration studies on the structural system complement elasticity studies on the materials of the structures [10-11]. This means that our work in this paper shall be useful.

The remainder of this paper consists of three sections. Section 2 presents mathematical models and methods. Section 3 provides numerical results and discussion. We conclude the paper in Section 4.

2. Mathematical models and methods

The vibration of a one-storey building can be modelled like a vibration problem of a spring-mass system, that is, to be a second-order ordinary differential equation. In this paper, we assume that vibration problems do not involve friction. This can be done for problems with negligible damping.

The mathematical model for the spring-mass problem without friction is

$$mx'' + kx = 0, \quad x \in \mathbb{R}^1, \tag{1}$$

with the vibration frequency $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. Here m represents the mass of the object and k denotes the stiffness of the pillar of the building. Furthermore, t is the time variable and x is the space variable. If the building has n levels, then the model is a large scale system of second order ordinary differential equations. Afterwards, the large scale system can be written in the form of matrix equations

$$MX'' + KX = 0, \quad X \in \mathbb{R}^n, \tag{2}$$

where M is the mass matrix, K is the stiffness matrix, and X is the vector of space variables. Before solving equation (2), we first review that equation (1) can be solved using existing numerical methods. The numerical methods that we consider in this paper are the finite difference and the Runge–Kutta types. The finite difference methods are the forward and central differences, and the Runge–Kutta methods are the Euler and Heun methods. By comparing these methods, we shall know which method has better performance than the others.

2.1. Case 1: If equation (1) is solved using forward difference method

Based on forward difference method algorithm, equation (1) can be changed to

$$m\frac{(x')_{t=t^{n+1}} - (x')_{t=t^n}}{\Delta t} + kx|_{t=t^n} = 0, \tag{3}$$

$$m\frac{\frac{x|_{t=t^{n+2}} - x|_{t=t^{n+1}}}{\Delta t} - \frac{x|_{t=t^{n+1}} - x|_{t=t^n}}{\Delta t}}{\Delta t} + kx|_{t=t^n} = 0, \tag{4}$$

$$m\frac{\frac{x^{n+2} - x^{n+1}}{\Delta t} - \frac{x^{n+1} - x^n}{\Delta t}}{\Delta t} + kx^n = 0, \tag{5}$$

$$m\frac{x^{n+2} - 2x^{n+1} + x^n}{\Delta t^2} + kx^n = 0, \tag{6}$$

or can be changed to

$$m\frac{x^{n+1} - 2x^n + x^{n-1}}{\Delta t^2} + kx^{n-1} = 0, \tag{7}$$

so that

$$m(x^{n+1} - 2x^n + x^{n-1}) = -kx^{n-1}\Delta t^2, \tag{8}$$

$$x^{n+1} - 2x^n + x^{n-1} = \frac{-kx^{n-1}\Delta t^2}{m}, \tag{9}$$

$$x^{n+1} = \frac{-kx^{n-1}\Delta t^2}{m} + 2x^n - x^{n-1}. \tag{10}$$

Equation (10) is the forward difference scheme for equation (1). Without loss of generality, we assume that the value of $k = 1$ and $m = 1$, then we obtain

$$x^{n+1} = -x^{n-1}\Delta t^2 + 2x^n - x^{n-1}. \tag{11}$$

2.2. *Case 2: If equation (1) is solved using central difference method*

Equation (1) can be changed to

$$m\frac{(x')_{t=t^{n+1}}-(x')_{t=t^{n-1}}}{2\Delta t}+kx|_{t=t^n}=0, \tag{12}$$

$$m\frac{\frac{x|_{t=t^{n+2}}-x|_{t=t^n}}{2\Delta t}-\frac{x|_{t=t^n}-x|_{t=t^{n-2}}}{2\Delta t}}{2\Delta t}+kx|_{t=t^n}=0, \tag{13}$$

$$m\frac{\frac{x^{n+2}-x^n}{2\Delta t}-\frac{x^n-x^{n-2}}{2\Delta t}}{2\Delta t}+kx^n=0, \tag{14}$$

$$m\frac{x^{n+2}-2x^n+x^{n-2}}{4\Delta t^2}+kx^n=0, \tag{15}$$

or can be written as

$$m\frac{x^{n+1}-2x^n+x^{n-1}}{\Delta t^2}+kx^n=0, \tag{16}$$

so we get

$$m(x^{n+1}-2x^n+x^{n-1})=-kx^n\Delta t^2, \tag{17}$$

$$x^{n+1}-2x^n+x^{n-1}=\frac{-kx^n\Delta t^2}{m}, \tag{18}$$

$$x^{n+1}=\frac{-kx^n\Delta t^2}{m}+2x^n-x^{n-1}. \tag{19}$$

Equation (19) is the central difference scheme for equation (1). For $k=1$ and $m=1$, we obtain

$$x^{n+1}=-x^n\Delta t^2+2x^n-x^{n-1}. \tag{20}$$

Notice that the central difference scheme (20) is not the same as the forward difference scheme (11).

2.3. *Case 3: If equation (1) is solved using the first order Runge–Kutta method (Euler method)*

Given equation (1) where the value of $k=1$ and $m=1$, we obtain

$$x''+x=0. \tag{21}$$

Furthermore, we derived a system of first order differential equations from equation (21). Let $x_1=x$ and $x_2=x'$, so the system of equations (21) becomes

$$x_1'=x'=x_2, \tag{22}$$

$$x_2'=x''=-x=-x_1. \tag{23}$$

The system of equations (22) and (23) can be solved using the Euler method as follows

$$X^{n+1}=X^n+f(t_n,X^n)\Delta t, \tag{24}$$

or can be written in a matrix equation

$$X^{n+1}=X^n+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}{x_1}^n\\ {x_2}^n\end{pmatrix}\Delta t. \tag{25}$$

2.4. Case 4: If equation (1) is solved using a second order Runge–Kutta method (Heun method)
The Heun method for the system of equations (22) and (23) is

$$X^{n+1} = X^n + \frac{f(t_n, X^n) + f(t_{n+1}, X^{n+1})}{2}\Delta t, \tag{26}$$

With $f(t_{n+1}, X^{n+1}) \approx f(t_{n+1}, \overline{x_{n+1}})$, where $\overline{x_{n+1}} = X^n + f(t_n, X^n)\Delta t$ or in other words $\overline{x_{n+1}} = X^{n+1}$ in the Euler method. Therefore, the algorithm for Heun method becomes

$$X^{n+1} = X^n + \frac{f(t_n, X^n) + f(t_{n+1}, \overline{x_{n+1}})}{2}\Delta t, \tag{27}$$

or it can be written in a matrix equation

$$X^{n+1} = X^n + \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} {x_1}^n \\ {x_2}^n \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} {x_1}^{n+1*} \\ {x_2}^{n+1*} \end{pmatrix}\right]\frac{\Delta t}{2}, \tag{28}$$

where ${x_1}^{n+1*}$ and ${x_2}^{n+1*}$ are obtained from the one-step Euler method.

3. Results of finite difference methods and Runge–Kutta methods
This section compares results of forward and central difference methods, as well as those of the Euler and Heun methods in solving equation (1). Simulations of the four methods are done using MATLAB, with values $0 \le t \le 10$ and $\Delta t = 0.5; 0.25; 0.125; 0.0625$. Based on the algorithm that is made in MATLAB, we present a graphic image showing the comparison results of the four methods together with analytical results.

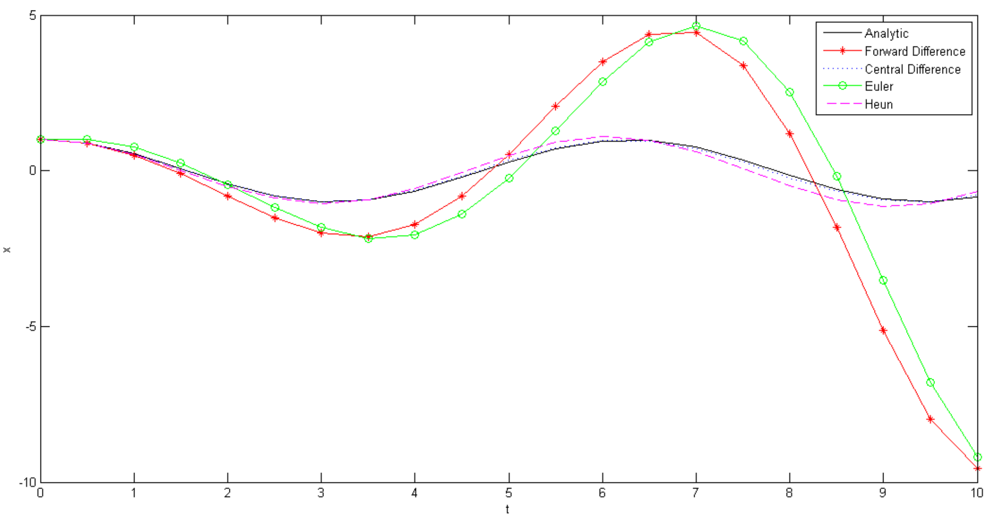

Figure 1. Analytical and numerical solutions using time step $\Delta t = 0.5$.

Representatives of our numerical results are shown in Figures 1-4. Respectively these figures show analytical solution together with numerical solutions using time steps $\Delta t = 0.5; 0.25; 0.125; 0.0625$. We observe that the central difference method and the Heun method approximate the analytical solution better than the forward difference method and the Euler method do.

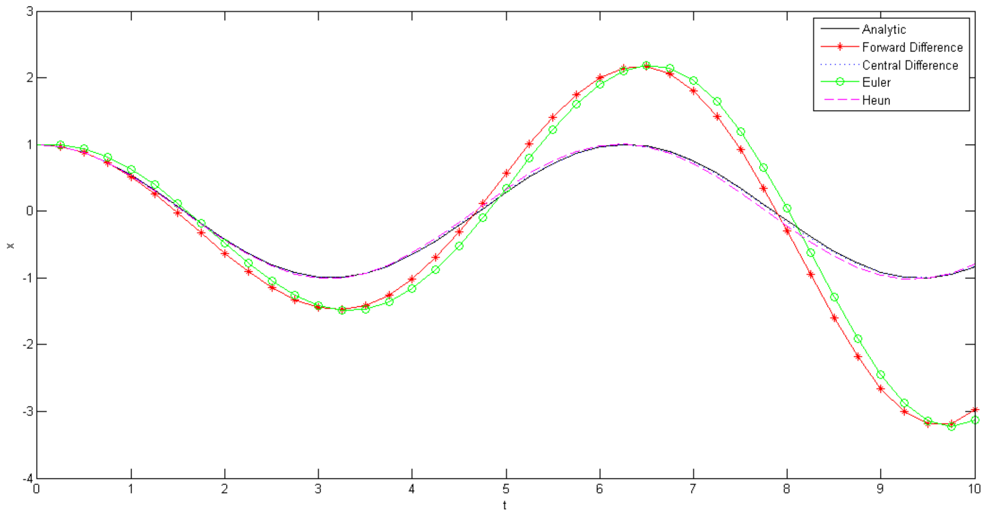

Figure 2. Analytical and numerical solutions using time step $\Delta t = 0.25$

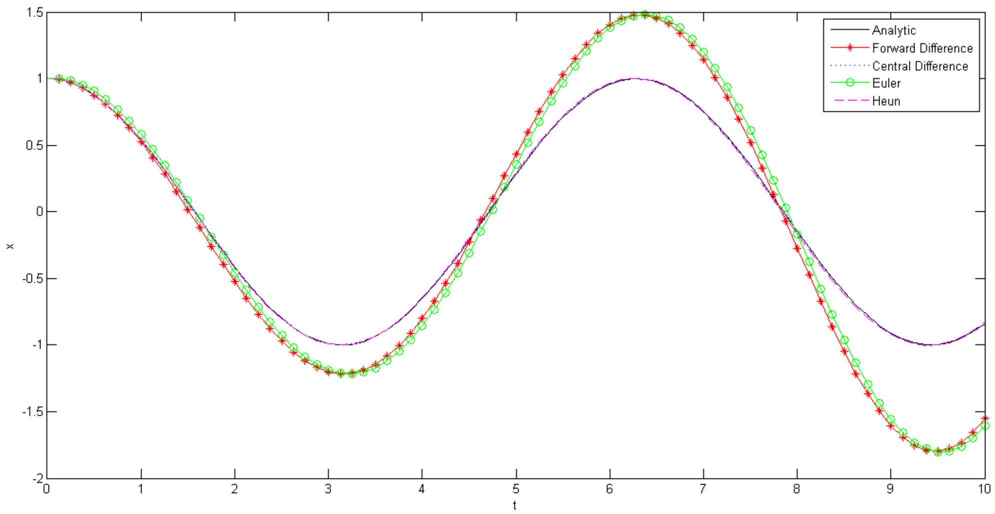

Figure 3. Analytical and numerical solutions using time step $\Delta t = 0.125$

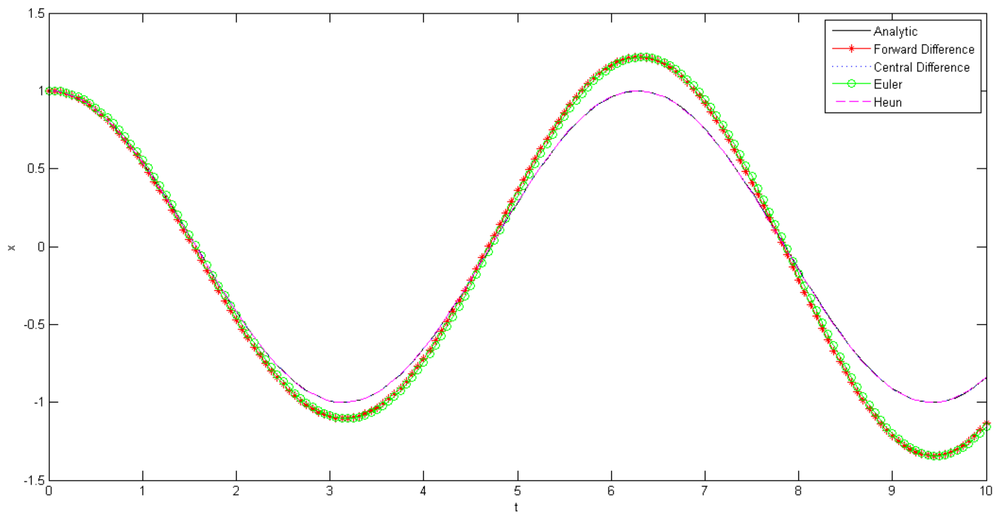

Figure 4. Analytical and numerical solutions using time step $\Delta t = 0.0625$

Table 1. Error comparison of the forward difference, central difference, Euler, and Heun methods.

Δt	Error			
	Forward difference	Central difference	Euler method	Heun method
0.5	1.9924	0.0291	1.8732	0.1284
0.25	0.6659	0.0074	0.6510	0.0313
0.125	0.2671	0.0019	0.2611	0.0078
0.0625	0.1199	0.0005	0.1169	0.0019

Furthermore, error comparison with varying time step is summarised in Table 1. We obtain that for central difference and the Heun methods, as the time step is halved, the error gets quartered. For forward difference and Euler methods, as the time step is halved, the error gets halved too. These mean that central difference and the Heun methods have the second order of accuracy. Forward difference and Euler methods have the first order of accuracy.

4. Conclusion

We have compared the performance of the forward finite difference, central finite difference, Euler, and Heun methods for solving vibration problems. We obtain that the central finite difference and the Heun methods have the second order of accuracy to solve vibration problems modelled in ordinary differential equations. These two methods produce more accurate solutions than the forward finite difference and the Euler methods having only the first order of accuracy.

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